PUPT-1708
hep-th/9707167

Strings and D-Branes at High Temperature

Sangmin Lee and Lárus Thorlacius

Joseph Henry Laboratories
Princeton University
Princeton, New Jersey 08544

Abstract

The thermodynamics of a gas of strings and D-branes near the Hagedorn transition is described by a coupled set of Boltzmann equations for weakly interacting open and closed long strings. The resulting distributions are dominated by the open string sector, indicating that D-branes grow to fill space at high temperature.

1 Introduction

Strings exhibit interesting behavior at high temperature. A gas of closed fundamental strings approaching the Hagedorn temperature T_H enters a long string phase, where any added energy goes into forming string rather than raising the temperature beyond T_H. In the long string phase the average number of long strings of a given length ℓ is proportional to $1/\ell$ and the average total number of long strings is $\log(E)$, where E is the total energy in the microcanonical ensemble. These results are readily found by solving a Boltzmann equation for long closed strings [1, 2]. They can also be obtained from the exponentially rising density of states for free strings, provided one is careful to properly define the microcanonical ensemble [1, 3, 4]. At very high energy density the description in terms of weakly coupled strings in flat spacetime breaks down and the physics of the system is unknown.

Strings are not the only extended objects in string theory and it is natural to ask how the above physical picture is changed when the system also contains p-branes. We will address this question in the context of Type II strings and D-branes, by extending the Boltzmann equation approach of ref. [2] to include the auxiliary open string sector that describes the D-brane dynamics. We find that the long closed string phase is suppressed in the presence of a dilute gas of D-branes (and anti-branes) and the system is instead dominated by long open strings as the Hagedorn temperature is approached.

The interactions of fundamental strings always include gravity and the thermodynamic limit, where one considers a fixed energy density as the volume becomes arbitrarily large, is not well defined for gravitational systems due to the Jeans instability. For a fixed value of the string coupling g we can only consider a thermal ensemble on finite length scales,

$$R^2 < \frac{1}{g^2 \rho}, \tag{1}$$

where ρ is the energy density. Alternatively, the limit of large volume can be taken for fixed ρ only if g is simultaneously scaled to zero fast enough.

This has important consequences if we wish to include D-branes in the system. A Dirichlet p-brane carries a conserved charge that couples to a Ramond-Ramond, rank p+1, antisymmetric gauge potential. A finite volume target space must carry zero net charge, and as the D-branes are the only objects in the theory that carry the R-R charge, we must have an equal number of branes and anti-branes in our system.[1] The annihilation of branes and anti-branes will deplete their numbers in a true equilibrium configuration unless the energy density is sufficiently high, $\rho \sim 1/g$, to have brane anti-brane pairs forming at a rapid rate. At such high energy densities, however, the dynamics of D-branes are unknown and our approximations break down.

We will instead consider a dilute collection of branes and anti-branes interacting with a gas of strings. This is not an equilibrium configuration but if the average time between

[1]Some of the p-branes may be in bound states with (p+2)-branes or (p+4)-branes but the net p-brane charge still has to be zero for all p.

brane anti-brane encounters is long compared to the time it takes the string gas to reach equilibrium, then the configurations at intermediate time scales will be well approximated by solutions to our transport equations. Note that we cannot take the D-branes to be too dilute. At any fixed value of the string coupling, the Jeans instability limits the size of the entire system and we want the average separation between branes to be small on that scale. As we shall see below, for weakly coupled strings there is a range of D-brane number densities where both of these requirements are satisfied.

2 Boltzmann equation for closed strings

We use Boltzmann transport equations to obtain the distribution of string lengths, both for the closed strings in the bulk of spacetime and the open string sector associated to the D-branes. Boltzmann equations for strings have previously been studied in a discrete model by Salomonson and Skagerstam [1], and in a continuum approach by Lowe and Thorlacius [2] which can be easily adapted to account for the two coupled string sectors.

To review, let us first consider the simpler system of only closed strings and no D-branes. The Boltzmann equation for weakly coupled closed strings is [2],

$$\begin{aligned}\frac{\partial f(\ell)}{\partial t} \;=\; & \frac{\kappa}{V}\Big\{-\frac{1}{2}\ell^2 f(\ell)+\frac{1}{2}\int_0^\ell d\ell'\;\ell'(\ell-\ell')f(\ell')f(\ell-\ell')-\ell f(\ell)\int_0^\infty d\ell'\ell' f(\ell') \\ & +\int_0^\infty d\ell'(\ell+\ell')f(\ell+\ell')\Big\}, \qquad (2)\end{aligned}$$

where $f(\ell)$ is the average number of strings of length ℓ, κ is a positive constant proportional to g^2, and V is the volume of the system.[2] The terms on the right hand side all involve the three closed string interaction shown in Figure 1a. Higher order interactions of four or more closed strings may be ignored at weak string coupling.

The first term corresponds to a loop of length ℓ self-intersecting and splitting into two strings of length ℓ' and $\ell-\ell'$, while the second term describes the reverse process where two smaller strings join to form a string of length ℓ. The third term is due to strings of length ℓ and ℓ' joining to form a single string of length $\ell+\ell'$, while the fourth term represents a string of length $\ell+\ell'$ splitting into strings of length ℓ and ℓ'.

In each of the four terms, two string segments meet and exchange ends at the intersection point. The interaction rate is proportional to the coefficient κ which involves a sum over relative momentum and orientation of the segments. In the long string limit this sum does not depend on string length nor on whether the two segments belong to separate strings or the same one, and thus the coefficient is the same for all four terms. The overall factor of $1/V$ reflects the fact that the string segments have to be at the same place in the embedding space in order for an interaction to take place.

At equilibrium we are interested in a static solution of the Boltzmann equation, where $\frac{\partial}{\partial t}f(\ell)=0$, and the terms on the right hand side of (2) cancel. The equilibrium solution is

[2] We have set $\alpha'=1$ for convenience.

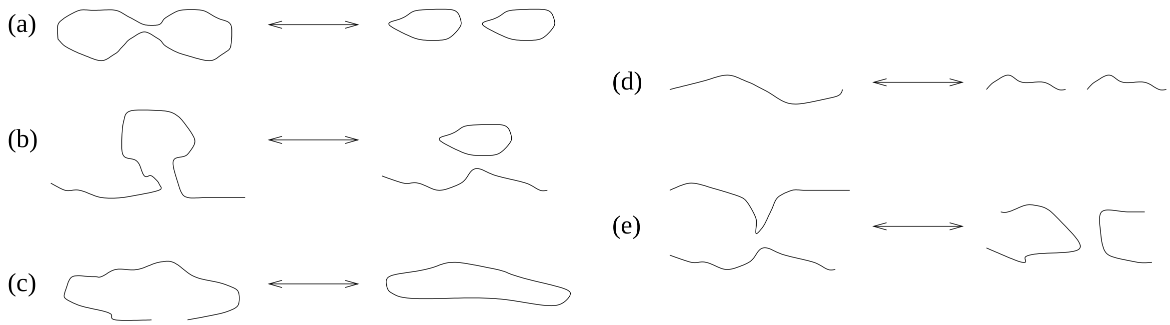

Figure 1: Leading order string interactions.

easily obtained [2],

$$f(\ell) = \frac{1}{\ell} e^{-\ell/L} , \tag{3}$$

where L is the average total length of string in the ensemble.

The energy of a long string is to a good approximation proportional to its length, $\varepsilon = \sigma\ell$, where σ is the string tension (σ is $O(1)$ in our units). The distribution $f(\ell)$ is related to the single-string density of states $\omega(\varepsilon)$ as follows:

$$f(\ell)\, d\ell = \omega(\varepsilon) e^{-\beta\varepsilon}\, d\varepsilon. \tag{4}$$

Inserting the equilibrium solution (3) we read off

$$\omega(\varepsilon) = \frac{e^{\beta_H \varepsilon}}{\varepsilon}, \tag{5}$$

where

$$\beta_H = \beta - \frac{1}{L} , \tag{6}$$

is identified as the inverse Hagedorn temperature. The corresponding multi-string density of states is

$$\Omega(E) = \exp(\beta_H E), \tag{7}$$

and the single-string distribution function $d(\varepsilon, E)$, which gives the average number of strings carrying energy between ε and $\varepsilon + d\varepsilon$ in a system of total energy E, is given by

$$d(\varepsilon, E) \approx \frac{\omega(\varepsilon)\Omega(E-\varepsilon)}{\Omega(E)} \approx \frac{1}{\varepsilon}. \tag{8}$$

This result is valid for $c < \varepsilon < E - c$ where c is some constant independent of E [4].

The average total number of long strings is then $\log E$. At an energy density of order one in string units the average total length of string in the gas is $L \sim V$ so the typical long string has length

$$\bar{\ell} \sim \frac{V}{\log V} , \tag{9}$$

which is large compared to the linear size of the system, $L_0 = V^{1/d}$.

3 Boltzmann equations for strings and D-branes

We now include a dilute gas of D-branes in the system. We consider parallel p-branes and anti-p-branes in a target space $\mathcal{M} = \mathbf{R} \otimes \mathcal{K}_\parallel \otimes \mathcal{K}_\perp$, where $\mathcal{K}_\parallel$ and $\mathcal{K}_\perp$ have dimension $d_\parallel = p$ and $d_\perp = 9 - p$ respectively. For simplicity, we will assume $0 \leq p \leq 6$. The branes are wrapped around $\mathcal{K}_\parallel$, which we will take to be compact with volume of order one in string units, but are free to move around in the transverse space $\mathcal{K}_\perp$. The transverse volume $V_\perp$ is macroscopic but bounded by the Jeans volume,

$$V_\perp < \left(\frac{1}{g^2\rho}\right)^{d_\perp/2} . \tag{10}$$

Let n be the number density of p-branes and anti-branes in the transverse space,

$$n = \frac{N_p + \bar{N}_p}{V_\perp} = \frac{2N_p}{V_\perp} . \tag{11}$$

The typical separation between branes is $l_p \sim n^{-1/d_\perp}$. We must take this distance to be large in string units for otherwise the energy density due to D-branes would no longer be small compared to $1/g$.

The string gas now contains both closed and open strings. Let $f(\ell)$ denote the average number of closed strings of length ℓ in the system and $p(\ell)$ the average number of open strings of length ℓ. There are two coupled Boltzmann equations which express the rate of change of $f(\ell)$ and $p(\ell)$ with time. For weakly interacting strings the terms in the Boltzmann equations correspond to the various string interactions shown in Fig. 1. Let us consider the $f(\ell)$ equation first,

$$\begin{aligned} \frac{\partial f(\ell)}{\partial t} \;=\; & \frac{\kappa}{V}\Big\{-\frac{1}{2}\ell^2 f(\ell) + \frac{1}{2}\int_0^\ell d\ell'\ell' f(\ell')(\ell-\ell')f(\ell-\ell') - \ell f(\ell)\int_0^\infty d\ell'\ell' f(\ell') \\ & + \int_0^\infty d\ell'(\ell+\ell')f(\ell+\ell')\Big\} + \frac{b}{2nV}p(\ell) - an\ell f(\ell) \\ & + \frac{\kappa}{V}\Big\{\int_\ell^\infty d\ell'(\ell'-\ell)p(\ell') - \ell f(\ell)\int_0^\infty d\ell'\ell' p(\ell')\Big\}. \end{aligned} \tag{12}$$

The first four terms on the right hand side correspond to the closed string interaction in Figure 1a and are the same as in the Boltzmann equation of the previous section. The remaining terms correspond to closed strings interacting with open strings attached to D-branes, as indicated in Figure 1b and 1c. A splitting interaction where a closed string is converted into an open string, or an open string splits into two open strings, comes with a factor a, which is proportional to g and includes a sum over the momentum and relative orientation of the string segments involved. The reverse joining process is characterized by another coefficient b, which is also proportional to g.

The equation for $p(\ell)$ receives contributions from the open string interactions in Figure 1d and 1e, and also from the mixed interactions in Figure 1b and 1c,

$$\frac{\partial p(\ell)}{\partial t} \;=\; \frac{b}{2nV}\int_0^\ell d\ell' p(\ell')p(\ell-\ell') - an\ell p(\ell) + 2an\int_\ell^\infty d\ell' p(\ell') - \frac{b}{nV}p(\ell)\int_0^\infty d\ell' p(\ell')$$

$$
\begin{aligned}
&+\frac{\kappa}{V}\Big\{\int_0^\ell d\ell_1 p(\ell_1)\int_{\ell-\ell_1}^{\ell} d\ell_2(\ell_1+\ell_2-\ell)p(\ell_2)+2\int_0^\ell d\ell_1\ell_1 p(\ell_1)\int_\ell^\infty d\ell_2 p(\ell_2)\\
&\quad+\ell\int_\ell^\infty d\ell_1 p(\ell_1)\int_\ell^\infty d\ell_2 p(\ell_2)-\ell p(\ell)\int_0^\infty d\ell'\ell' p(\ell')\Big\}\\
&+\frac{\kappa}{V}\Big\{\int_0^\ell d\ell'\ell' f(\ell')(\ell-\ell')p(\ell-\ell')-\frac{1}{2}\ell^2 p(\ell)+\ell\int_\ell^\infty d\ell' p(\ell')\\
&\quad-\ell p(\ell)\int_0^\infty d\ell'\ell' f(\ell')\Big\}+an\ell f(\ell)-\frac{b}{2nV}p(\ell).
\end{aligned}
\tag{13}
$$

At equilibrium, the distribution of strings does not change with time,

$$
\frac{\partial f}{\partial t}=\frac{\partial p}{\partial t}=0. \tag{14}
$$

The coupled equations (12) and (13) appear considerably more complicated than the Boltzmann equation (2) for closed strings. One nevertheless finds a simple equilibrium solution,

$$
f(\ell)=\frac{1}{\ell}\exp[-\ell/L_c],\qquad p(\ell)=\frac{N_o}{L_c}\exp[-\ell/L_c]. \tag{15}
$$

The distributions are characterized by the average total length of closed and open string and the average total number of open strings,

$$
L_c\equiv\int_0^\infty d\ell\,\ell f(\ell),\qquad L_o\equiv\int_0^\infty d\ell\,\ell p(\ell),\qquad N_o\equiv\int_0^\infty d\ell\,p(\ell). \tag{16}
$$

These parameters satisfy the relations

$$
\frac{N_o}{L_c}=\frac{2an^2V}{b},\qquad N_oL_c=L_o. \tag{17}
$$

It immediately follows that the open strings tend to dominate the ensemble. Whenever there is a macroscopic number of open strings present, $N_o >> 1$, the total length of closed string will be vanishing compared to the total length of open string, $L_c << L_o$. In other words, the D-branes efficiently chop up the strings so that the closed string gas is replaced by a web of open strings attached to the D-branes.

The typical length of a long open string can be obtained from the single string distribution function for open strings $d_o(\varepsilon, E)$ in the microcanonical ensemble. The first step is to read off the single-string density of states for open strings from the equilibrium solution (15),

$$
\omega_o(\varepsilon)=\frac{2an^2V}{b\sigma}\exp\left(\beta_H\varepsilon\right). \tag{18}
$$

The corresponding multi-string density of states can be obtained in a saddle point approximation, which is valid when the total number of D-branes in the gas is large,

$$
\Omega_o(E)\approx\exp\Big\{\sqrt{\frac{8an^2VE}{b\sigma}}+\beta_H E\Big\}. \tag{19}
$$

The open string distribution function is then

$$
\begin{aligned}
d_o(\varepsilon, E) &\approx \frac{\omega_o(\varepsilon)\Omega_o(E-\varepsilon)}{\Omega_o(E)} \\
&\sim \exp\Big\{-\sqrt{\frac{8an^2VE}{b\sigma}}(1-\sqrt{1-\frac{\varepsilon}{E}})\Big\}.
\end{aligned} \tag{20}
$$

The distribution function has a characteristic energy scale,

$$
\varepsilon_0 = \frac{1}{n}\sqrt{\frac{b\rho}{a\sigma}}, \tag{21}
$$

below which the distribution is more or less flat and above which it is strongly suppressed. The energy of a long string is proportional to its length so there is a corresponding characteristic length scale for the open strings in the gas. Since the rate coefficients a and b are both proportional to g the characteristic length is given in string units by

$$
l_0 \sim \frac{\rho^{1/2}}{n}, \tag{22}
$$

up to factors of order one. At the Hagedorn energy density, $\rho \sim 1$, the characteristic open string length is long compared to the typical D-brane separation $l_p \sim n^{-1/d_\perp}$, so the D-branes are effectively overlapping.

Our calculations have involved a number of assumptions and we close this section with a discussion of their range of validity. The Boltzmann equations only have the relatively simple form of (12) and (13) if we can assume that points on a string that are separated by a finite parameter distance are at uncorrelated positions in the embedding space. This criterion is satisfied if the volume V_s occupied by a long string when placed in an infinite space is large compared to the system volume V. The embedding of a long string of length ℓ is a random walk which occupies a volume $V_s \sim \ell^{d_\perp/2}$ in $d_\perp$ spatial dimensions. Let us assume that the energy density in the system is order one in string units and that the spatial volume is close to the maximum value allowed before the Jeans instability leads to gravitational collapse, $V \sim g^{-d_\perp}$. The characteristic open string length (22) involves the D-brane number density and we find that $V_s > V$ is satisfied only if

$$
n < g^2. \tag{23}
$$

We also want to impose the condition that the typical D-brane separation, $l_p \sim n^{-1/d_\perp}$, is small compared to the linear size of the system, which requires $n > g^{d_\perp}$. Since $d_\perp \geq 3$ for p-branes with $p \leq 6$ there is always an allowed range for the number density.

If V_s for typical strings is smaller than the system volume the terms in the Boltzmann equations that correspond to string self-interactions have to be modified by replacing the factor of V in the denominator by V_s. We have not found explicit solutions of the resulting equations but we expect such a modification to increase the weight of shorter strings in the distribution when the D-brane number density is higher than in (23).

Our results also rest on the assumption that the strings take less time to reach an equilibrium configuration than the timescale on which the D-branes and anti-branes meet and annihilate. It turns out that this is a weaker condition than (23). The D-brane lifetime can be estimated as

$$\tau \sim \frac{1}{n\sigma v}, \tag{24}$$

where σ is the brane anti-brane annihilation cross section and v is the average D-brane speed. The D-branes are immersed in a string gas with an energy density of order one in string units. Their kinetic energy will therefore be of order one which means that their motion is non-relativistic with $v \sim g^{1/2}$. The eikonal approximation to the low velocity scattering amplitude of a D-brane and anti-brane pair develops a tachyonic instability if the impact parameter is less than a certain value of order one in string units [5]. We use this as an estimate for the annihilation cross section and take $\sigma \sim O(1)$. The D-brane lifetime is then found to be $\tau \sim g^{-1/2}n^{-1}$.

This is to be compared to the time it takes the string distribution to find equilibrium. For an ensemble dominated by open strings the relevant processes are the open string interactions in Figure 1d and 1e and we can ignore the closed strings. In the long string limit the probability for a pair of string segments, that happen to come across each other, to interact is independent of the overall string lengths. Furthermore, if a long string undergoes an interaction its new length will be largely uncorrelated to the old one. It follows that a gas of long strings will find equilibrium on a time scale where the strings have interacted a few times each. It turns out that the exchange interaction Figure 1e, which occurs in the bulk of spacetime, is more efficient at redistributing length among the open strings than the endpoint interaction in Figure 1d, which can only take place inside a D-brane worldvolume. The probability for a particular string to participate in such an exchange interaction in a unit of time goes as $g^2\ell$, where ℓ is the length of the string. Using the characteristic length (22), and assuming $\rho \sim 1$, the time to reach equilibrium is

$$\tau_e \sim \frac{n}{g^2}. \tag{25}$$

It is short compared to the D-brane lifetime if the D-brane number density satisfies $n < g^{3/4}$, which is a weaker condition than (23). The equilibrium time scale τ_e can also be estimated by linearizing the long string Boltzmann equation (13) around the equilibrium solution (15) and find the rate at which the perturbation decays. This second method gives an answer that agrees with (25).

4 Discussion

D-branes strongly affect the long string phase associated with the Hagedorn transition. In the absence of D-branes it consists of long closed strings that traverse the entire system many times over, but a dilute collection of D-branes will suppress the long closed strings and

replace them with long open strings that are attached to the D-branes. The typical open string is long compared to the D-brane separation and is therefore most likely to have its ends on different D-branes. Since open strings are associated to the dynamics of the D-branes themselves these results suggest a physical picture where D-branes grow with temperature as the Hagedorn transition is approached and eventually overlap to fill space.

The system enters the long string phase when the energy density becomes of order one in string units. For a D-brane number density that satisfies the condition (23) the 'bare' energy associated with the D-branes is then small compared to the energy carried in the open strings attached to them. If the energy density of the system is increased further the characteristic open string length increases and the D-branes become more interconnected. If the energy density becomes of order $1/g$ non-perturbative effects, such as pair creation of branes and anti-branes, become important and our description in terms of weakly interacting strings breaks down.

In this paper we took the string coupling to be weak and limited the system size in order to avoid the Jeans instability but it would be interesting to incorporate long range gravitational effects into the description of the string distributions. Some related issues are addressed in recent work of Horowitz and Polchinski [6].

Acknowledgements

We are grateful to C. Callan, I. Klebanov, A. Peet and S.-J. Rey for useful discussions. This work was supported in part by a US Department of Energy Outstanding Junior Investigator Award under grant DE-FG02-91ER40671.

References

[1] P. Salomonson and B-S. Skagerstam, Nucl. Phys. **B268**, 349 (1986); Physica A **158**, 499 (1989).

[2] D. A. Lowe and L. Thorlacius, Phys. Rev. **D51**, 665 (1995).

[3] R. Brandenberger and C. Vafa, Nucl. Phys. **B316**, 391 (1989).

[4] N. Deo, S. Jain, and C-I. Tan, Phys. Rev. **D40**, 2626 (1989); Phys. Lett. **B220**, 125 (1989); N. Deo, S. Jain, O. Narayan, and C. Tan, Phys. Rev. **D45**, 3641 (1992).

[5] T. Banks and L. Susskind, *Brane – Anti-Brane Forces*, hep-th/9511194; G. Lifschytz, Phys. Lett. **B388**, 720 (1996).

[6] G. Horowitz and J. Polchinski, *A Correspondence Principle for Black Holes and Strings*, hep-th/9612146, G. Horowitz, talk at Strings 97, Amsterdam, June 16-21, 1997.